RECIPROCAL RELATIONS FOR THE DIFFERENTIAL OPERATORS OF THE THEORY OF ELASTICITY

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A derivation is presented of expressions which relate the differential operators of the theory of elasticity by virtue of their potential character. The corresponding relations for nonlinear systems with a finite number of degrees of freedom were apparently first obtained by A.R. Rzhanitsyn in 1952. He also indicated at that time the possibility of generalizing to the case of continuous systems. The fact that a complete system of potentials of the theory of elasticity and plasticity exists (established by Gol'denblat [1]) serves as the starting point for this paper.

1. For conservative, discrete systems with p degrees of freedom, a proof of the existence of functions U_l with the following properties and a method for constructing them are given in [1]:

$$-\partial U_l/\partial T_i = L_i(T_1,\dots T_l, V_{l+1},\dots V_p) = V_i \qquad (i = 1, 2,\dots l)$$

$$\partial U_l/\partial V_m = L_m(T_1,\dots T_l, V_{l+1},\dots V_p) = T_m \qquad (m = l+1, l+2,\dots p) \quad (1.1)$$

Here the T_i are generalized internal forces; V_m are the generalized displacements; and L_k are the algebraic operators which form the system of equations of the problem under consideration. In addition to the unknown quantities T_i and V_m, parameters occur in these equations which correspond to the geometric and elastic characteristics of the problem and also to the generalized external forces.

The set of all possible functions U_l possessing the properties indicated forms the complete system of potentials of the elastic system.

2. In the case of a continuous mechanical system a potential U_l becomes a functional which depends not only on the generalized forces and displacements, but also on their derivatives with respect to the coordinates. In order to explain the essentials of the matter it is sufficient to consider the following very simple functional in a two-dimensional region Ω:

$$U = \int_\Omega L(T, V)\, d\Omega = \int_\Omega F(T, V, T_x, T_y, V_x, V_y)\, dx\, dy \qquad \left(T_x = \frac{\partial T}{\partial x}, \dots\right) \quad (2.1)$$

Here $T \in M$, $V \in N$, where M and N are the sets of those functions in the Hilbert space $L_2(\Omega)$ which in the closed region Ω have continuous first derivatives and square-summable second derivatives and also satisfy the boundary conditions on the curve S bounding the region Ω. The set of pairs (T, V), which we shall regard as elements of the direct product of the space $L_2(\Omega)$ with itself, forms the region $D(L)$ of definition of the operator L and the functional U. The function F is assumed to be continuous along with its first and second derivatives with respect to any of its arguments.

Let us compute the partial variations of the functional U corresponding to (1.1), for arbitrary fixed $T, V \in D(L)$

$$\delta_T U = \int_\Omega \frac{\partial L}{\partial T}\, t\, d\Omega = \int_\Omega (F_T t + F_{T_x} t_x + F_{T_y} t_y)\, dx\, dy =$$

$$= \int_S (F_{T_x} dy - F_{T_y} dx)\, t + \int_\Omega \left(F_T - \frac{\partial}{\partial x} F_{T_x} - \frac{\partial}{\partial y} F_{T_y}\right) t\, dx\, dy = \int_\Omega V^0 t\, dx\, dy \quad (2.2)$$

$$\delta_V U = \int\limits_\Omega \frac{\partial L}{\partial V} v \, d\Omega = \int\limits_\Omega (F_V v + F_{V_x} v_x + F_{V_y} v_y) \, dx \, dy =$$

$$= \int\limits_S (F_{V_x} dy - F_{V_y} dx) v + \int\limits_\Omega \left(F_V - \frac{\partial}{\partial x} F_{V_x} - \frac{\partial}{\partial y} F_{V_y} \right) v \, dx \, dy = \int\limits_\Omega T^0 v \, dx \, dy \quad (2.3)$$

where the partial derivatives $\partial L / \partial T$ and $\partial L / \partial V$ are linear operators on $t \in M$ and $v \in N$ at the point (T, V).

As a result of the Lagrange transformation each of the variations acquires the form of the sum of a line integral and an integral over the region. The integrands of the integrals over the region are the left sides of the differential equations of the problem at hand. The natural boundary conditions follow from the condition that the line integrals go to zero. In conformity with Eqs. (1.1), the system of differential equations of the problem may be represented in the form

$$L_1(T, V) = -\left(F_T - \frac{\partial}{\partial x} F_{T_x} - \frac{\partial}{\partial y} F_{T_y} \right) = V^0, \; L_2(T, V) = F_V - \frac{\partial}{\partial x} F_{V_x} - \frac{\partial}{\partial y} F_{V_y} = T^0 \quad (2.4)$$

3. Let us multiply the left-hand side of the first of Eqs. (2.4) by t and integrate over the region Ω. Then, taking account of the boundary conditions, we carry out the inverse of the Lagrange transformation of Eq. (2.2) and write out the integrand obtained. Denoting the operation which has been performed by an asterisk on the corresponding operator, we have, obviously

$$L_1^* t = -(F_T t + F_{T_x} t_x + F_{T_y} t_y) \equiv -\frac{\partial L}{\partial T} t \quad (3.1)$$

We now perform the analogous operation on the second of Eqs. (2.4) (multiplication by v and subsequent transformation)

$$L_2^* v = F_V v + F_{V_x} v_x + F_{V_y} v_y \equiv \frac{\partial L}{\partial V} v \quad (3.2)$$

In accordance with the assumption adopted above concerning the function F in (2.1) for all $(T, V) \in D(L)$, the operator L has a continuous second differential on the set (T, V) and [2]

$$\frac{\partial^2 L}{\partial T \, \partial V} tv = \frac{\partial^2 L}{\partial V \, \partial T} vt \quad (3.3)$$

Taking account of this equation, we obtain from (3.1) and (3.2)

$$\frac{\partial L_1^*}{\partial V} tv = -\frac{\partial L_2^*}{\partial T} vt \quad (3.4)$$

It is now easy, by analogy based on (1.1), to write down the relations of similar type for a continuous system with l unknown generalized forces T_i and $(p - l)$ generalized displacements V_m. It differs from the preceding only in that: (1) an element of $(T_1, \ldots T_l, V_{l+1}, \ldots V_p)$ of the region of definition of the functional U_l is an element of the product of p Hilbert spaces; (2) the functions $T_1, \ldots V_p$ must have in the region Ω derivatives of whatever order is required by the problem under consideration; and (3) the region Ω need not necessarily be two-dimensional.

Considering, as before, that the operator in the integrand of the functional U_l is twice continuously differentiable, we obtain

$$\frac{\partial L_i^*}{\partial T_j} t_i t_j = \frac{\partial L_j^*}{\partial T_i} t_j t_i, \qquad \frac{\partial L_m^*}{\partial V_n} v_m v_n = \frac{\partial L_n^*}{\partial V_m} v_n v_m, \qquad \frac{\partial L_i^*}{\partial V_m} t_i v_m = -\frac{\partial L_m^*}{\partial T_i} v_m t_i \quad (3.5)$$

$$(i, j = 1, 2, \ldots l; \; m, n = l+1, l+2, \ldots p)$$

These are the relations which have been sought between operators for any conservative mechanical system. They can be used to verify the equations obtained from consideration of the statical, geometric and physical aspects of a problem.

For discrete systems the differential operators become algebraic ones and instead of (3.5) we have

$$\frac{\partial L_i}{\partial T_j} = \frac{\partial L_j}{\partial T_i}, \qquad \frac{\partial L_m}{\partial V_n} = \frac{\partial L_n}{\partial V_m}, \qquad \frac{\partial L_i}{\partial V_m} = -\frac{\partial L_m}{\partial T_i} \quad (3.6)$$

If the discrete system is linear, Eqs. (3.6), as can easily be seen, become the well-known relations between the coefficients of the flexibility method, stiffness method and mixed method of structural mechanics

$$\delta_{ij} = \delta_{ji}, \; r_{mn} = r_{nm}, \qquad r_{im} = -\delta_{mi} \tag{3.7}$$

Thus, Eqs. (3.7) are of the same nature as the relations (3.5) which generalize them. Therefore, (3.5) can also be considered as reciprocal relations.

4. As an example we shall apply the relations which have been found to von Kármán's equations for large deflections of a plate

$$L_1(\Phi, W) = \frac{1}{Eh}\nabla^4\Phi + W_{xx}W_{yy} - W_{xy}^2 = 0$$

$$L_2(\Phi, W) = D\nabla^4 W - \Phi_{xx}W_{yy} - \Phi_{yy}W_{xx} + 2\Phi_{xy}W_{xy} = q \tag{4.1}$$

Multiplying the first equation by φ and the second by w, we obtain after two integrations by parts, having due regard to the boundary conditions,

$$L_1{}^*\varphi = \frac{1}{Eh}\varphi\nabla^4\Phi + \frac{1}{2}(W_x{}^2\varphi_{yy} + W_y{}^2\varphi_{xx}) + W_xW_y\varphi_{xy} \tag{4.2}$$

$$L_2{}^*w = \Phi_{xx}W_yw_y + \Phi_{yy}W_xw_x - \Phi_{xy}(W_xw_y + W_yw_x) + D\nabla^4 Ww \tag{4.3}$$

In the calculations of (4.3) the conditions of equilibrium in the tangential directions were taken into account. One more differentiation of the expressions which have been found gives

$$\frac{\partial L_1{}^*}{\partial W}\varphi w = -W_xw_x\varphi_{yy} - W_yw_y\varphi_{xx} + (W_xw_y + W_yw_x)\varphi_{xy} = -\frac{\partial L_2{}^*}{\partial\Phi}w\varphi \tag{4.4}$$

As is apparent from this example, the use of the reciprocal relations (3.5) does not, in general, assume knowledge of the corresponding potentials.

As is well known, in using the variational methods of Ritz and of Bubnov and Galerkin, a continuous system becomes a system with a finite number of degrees of freedom in the computational scheme. Therefore, the algebraic equations which are obtained must satisfy Eqs. (3.6). In the application of the Kantorovich and Vlasov methods, the differential equations which are obtained after lowering the number of independent variables must satisfy Eqs. (3.5), as must the original equations.

BIBLIOGRAPHY

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